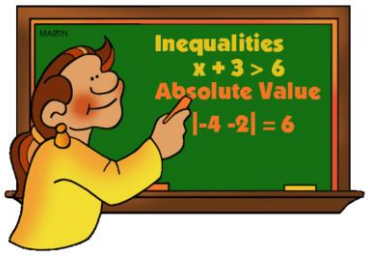
ABSOLUTE-VALUE INEQUALITIES

For linear inequalities (like $7n + 5 < 2$), the Basic Principle of Inequalities is sufficient (If you multiply or divide by a negative number, switch the inequality sign). But what about an inequality like



$$|3x - 4| + 8 \leq 20?$$

We will use the Boundary Point Method, discussed in the chapter Quadratic Inequalities.

❑ *THE BOUNDARY POINT METHOD*

1) Change the inequality to an *equation* and solve the equation.

2) The solutions of that equation are called the *boundary points*.

3) Plot the boundary points on the real line. This will create one or more *intervals* on the line.

4) Now choose a *test point* from each interval (any convenient number inside the interval) and plug it into the original inequality. If the test point satisfies the inequality, then the entire interval in which the test point lies is a part of the solution of the inequality. If the test point does not satisfy the inequality, then the associated interval is not part of the solution.

5) Lastly, check the boundary points themselves in the inequality.

❑ *ABSOLUTE VALUE INEQUALITIES*

EXAMPLE 1: **Solve for x: $|2x - 6| - 8 \geq 0$**

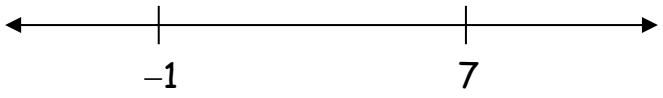
Solution: We do this by following the five steps of the Boundary Point Method.

1. Solve the associated equation:

$$|2x - 6| - 8 = 0$$
$$\Rightarrow \quad |2x - 6| = 8$$
$$\Rightarrow \quad 2x - 6 = 8 \quad \text{OR} \quad 2x - 6 = -8$$
$$\Rightarrow \quad 2x = 14 \quad \text{OR} \quad 2x = -2$$
$$\Rightarrow \quad \mathbf{x = 7 \quad OR \quad x = -1}$$

2. The boundary points are therefore 7 and –1.

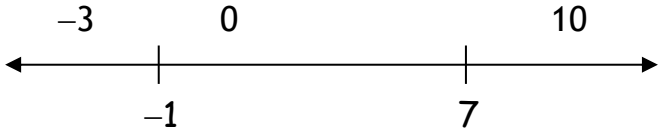
3.



Plotting the two boundary points produces three intervals:

$(-\infty, -1)$ $(-1, 7)$ $(7, \infty)$

4.



We have chosen three test points: **–3**, **0**, and **10**, one from each interval. We now test each point in the original inequality, $|2x - 6| - 8 \geq 0$:

$\lvert 2(\mathbf{-3}) - 6\rvert - 8 \geq 0$	$\lvert 2(\mathbf{0}) - 6\rvert - 8 \geq 0$	$\lvert 2(\mathbf{10}) - 6\rvert - 8 \geq 0$
$\lvert -6 - 6\rvert - 8 \geq 0$	$\lvert 0 - 6\rvert - 8 \geq 0$	$\lvert 20 - 6\rvert - 8 \geq 0$
$\lvert -12\rvert - 8 \geq 0$	$\lvert -6\rvert - 8 \geq 0$	$\lvert 14\rvert - 8 \geq 0$
$12 - 8 \geq 0$	$6 - 8 \geq 0$	$14 - 8 \geq 0$
$4 \geq 0$	$-2 \geq 0$	$6 \geq 0$
True	False	True
$(-\infty, 1)$ <u>is</u> part of the solution.	$(-1, 7)$ is <u>not</u> part of the solution.	$(7, \infty)$ <u>is</u> part of the solution.

5. Last, we check the boundary points themselves.

   If you substitute -1 into the inequality $\lvert 2x - 6\rvert - 8 \geq 0$, it should work out to $0 \geq 0$, which is true. Thus, -1 <u>is</u> part of the solution. Similarly, checking the boundary point 7 will also result in $0 \geq 0$, so 7 <u>is</u> part of the solution.

Our final answer comprises the two intervals from Step 4 and the two boundary points confirmed in Step 5.

$$\boxed{(-\infty, -1] \cup [7, \infty)}$$

In English, this means that any number which is either ≤ -1 **or** ≥ 7 will work in the original inequality.

<u>EXAMPLE 2:</u> **Solve for n: $2 - \lvert 8 - 2n\rvert > 0$**

<u>Solution:</u>

1. Solve the associated equation:

$$\begin{aligned} & 2 - \lvert 8 - 2n\rvert = 0 \\ \Rightarrow \quad & -\lvert 8 - 2n\rvert = -2 && \text{(subtract 2 from each side)} \\ \Rightarrow \quad & \lvert 8 - 2n\rvert = 2 && \text{(multiply each side by } -1\text{)} \end{aligned}$$

$\Rightarrow \quad 8 - 2n = 2 \quad \text{OR} \quad 8 - 2n = -2$

$\Rightarrow \quad n = 3 \quad \text{OR} \quad n = 5$

2. The boundary points are **3** and **5**.

3. The three intervals are therefore $(-\infty, 3)$, $(3, 5)$, and $(5, \infty)$.

4. Choose the test point $0 \in (-\infty, 3)$. Plugging 0 into the inequality $2 - |8 - 2n| > 0$ gives $-6 > 0$. Therefore, $(-\infty, 3)$ is <u>not</u> part of the answer.

   Choose the test point $4 \in (3, 5)$. Plugging 4 into the inequality $2 - |8 - 2n| > 0$ gives $2 > 0$. Thus, $(3, 5)$ <u>is</u> part of the answer.

   Choose the test point $9 \in (5, \infty)$. Plugging 9 into the inequality $2 - |8 - 2n| > 0$ gives $-8 > 0$, which implies that $(5, \infty)$ is <u>not</u> part of the answer.

5. You check the boundary points, 3 and 5, in the inequality. Neither of them works.

Our final answer consists of the middle interval <u>without</u> the boundary points. This can be written $\{n \in \mathbb{R} \mid 3 < n < 5\}$, or,

$$\boxed{(3, 5)}$$

Homework

Solve each absolute value inequality:

1. $|3a - 7| - 4 \geq 0$
2. $5 - |3 - c| > 0$
3. $-2 + |4y - 6| \leq 0$
4. $|3x - 9| - 18 \geq 0$

5. $|w + 7| \leq 10$

6. $|a + 1| - 4 > 0$

Review Problems

Solve each inequality:

7. $2(x - 5) - 9(2x + 1) > 0$

8. a. $|3x - 15| \geq 30$
   b. $|x - 5| > 0$
   c. $|9 - 3x| \leq 0$
   d. $|2x + 6| \geq 0$
   e. $|3 - x| < 0$
   f. $2|3x - 12| + 8 < 20$

9. $\frac{x}{a} > c$ Hint: We know a can't be 0 in the problem, but do we know for sure whether $a > 0$ or $a < 0$?

10. $|4x + 12| < 20$

11. True/False:

    a. When solving a linear inequality, one needs to reverse the inequality symbol only when dividing each side by a negative number.

    b. The solution of the inequality $ax \leq b$ is $x \leq \frac{b}{a}$.

    c. The solution of $|2g - 6| \geq 8$ is $(-\infty, -1] \cup [7, \infty)$.

    d. The solution of $|7 - 3x| < 0$ is $\mathbb{R}$.

    e. The solution of $|8y + 17| \geq 0$ is $\mathbb{R}$.

    f. The solution of $|x - 10| < 25$ is $[-15, 35]$.

Solutions

1. $(-\infty, 1] \cup \left[\frac{11}{3}, \infty\right)$ **2**. $(-2, 8)$ **3**. $[1, 2]$

4. $(-\infty, -3] \cup [9, \infty)$ **5**. $[-17, 3]$ **6**. $(-\infty, -5) \cup (3, \infty)$

7. $\left(-\infty, -\frac{19}{16}\right)$

8. a. $(-\infty, -5] \cup [15, \infty)$ b. $(-\infty, 5) \cup (5, \infty)$; or $\mathbb{R} - \{5\}$

c. $\{3\}$ d. $\mathbb{R}$ e. $\varnothing$ f. $(2, 6)$

9. Not solvable — it depends on whether a is positive or negative.

10. $(-8, 2)$

11. a. F b. F c. T d. F e. T f. F

"Education breeds confidence.
Confidence breeds hope.
Hope breeds peace."

Confucius